Int J Theor Phys (2007) 46: 2967–2975
DOI 10.1007/s10773-007-9411-5

Modified Heisenbeg Model and the Structure of Its Energy Spectrum

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Published online: 11 May 2007

Abstract In this paper, we show the interesting structure of energy spectrum for a modified Heisenberg XX model, explicit examples for low dimensions $N = 2, 3, \ldots, 6$ are also provided.

Keywords Modified Heisenbeger model · Energy spectrum

1 Introduction

The Heisenberg spin chain (HSC) is certainly one of the most important models in statistical mechanics. The model describes nearest-neighbor interacting spins, situated on the sites of a lattice N. The Heisenberg XXX model is exactly solvable by the Bethe ansatz [1]. Despite the simple form of the model, it bellies the rich physical behavior that it displays, and an understanding of the physics of the HSC in one-dimension has proved a formidable task for theoretical and mathematical physicists over the last six decades [2–11]. Recently, Heisenberg model has a renewed interest in quantum computation and quantum information due to the recent-discovered importance of entanglement in quantum theory. There has been much work on the implementation of quantum processing on solid state devices. An interesting type of entanglement, thermal entanglement, was studied in the context of the Heisenberg XXX [12, 13], XX [14], and XXZ [15] models. The Heisenberg model has been shown to have the potentiality to be used as a model for spin-spin interaction in a solid state quantum computer [16, 17]. It has been partially realized in quantum dots [16, 17], nuclear spins [18], and optical lattices [19]. Imamoglu et al. [20] have realized quantum information processing using quantum dot spins and cavity QED, and obtained an effective interaction Hamiltonian

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based on the XY spin chain between two quantum dots. The effective Hamiltonian was shown to be capable of constructing the Controlled-Not gate [20]. The XY Hamiltonian is given by

$$H=\sum_{n=1}^{N}(J_1 S_n^x S_{n+1}^x + J_2 S_n^y S_{n+1}^y) \tag{1}$$

where $S^i=\sigma^i/2$ $(i=x,y,z)$ and σ^i are Pauli operators. When $J_1=J_2$, the XY model becomes XX model.

Most recently, based on an interesting modified Heisenberg XX model, Christandl et al. [21] have proposed a class of qubit networks that admit perfect transfer of any quantum state over any distance in a fixed period of time. For the usual Heisenberg model, the perfect state transfer from one end of the chain to another is possible only for $N=2$ and $N=3$, while for the modified Heisenberg XX model, it works for arbitrary N. Additionally, in our previous work [22], we investigated the quantum nonlocality of this modified XX model based on Bell inequalities. The structure of energy spectrum for a multi-spin system is always an interesting and significant problem. To our knowledge, few people have studied the energy spectrum of the modified XX model. The purpose of this work is to address the problem, we find that such a model possesses a very interesting energy structure. This paper is organized as follows. In Sect. 2, we give a brief review for the modified Heisenberg XX model and show how it can realize a perfect state transfer. In Sect. 3, the energy structure of the modified model is presented and explicit examples for $N=2,3,\ldots,6$ are also provided. Discussion and conclusion are made in the last section.

2 Brief Review of the Modified XX Model

Let us consider the Hamiltonian of a general XX model

$$\begin{aligned}H&=\sum_{n=1}^{N-1}J_{n,n+1}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y)\\&=\frac{1}{2}\sum_{n=1}^{N-1}J_{n,n+1}(\sigma_n^+\sigma_{n+1}^-+\sigma_n^-\sigma_{n+1}^+)\end{aligned} \tag{2}$$

where $J_{n,n+1}$ are coupling strengths between lattices n and $n+1$, and $\sigma_n^{\pm}=\sigma_n^x\pm i\sigma_n^y$. Obviously, the Hamiltonian H describes a nearest-neighbor interaction spin chain. Hamiltonian H has 2^N complete and orthogonal eigenvectors, which span the Hilbert space of H. The Hilbert space of H can be divided into $N+1$ subspaces. The first subspace has only one (because $C_N^0=1$) eigenvector with zero-value of eigenvalue, i.e.,

$$|\psi_0\rangle=|00\cdots0\rangle,\qquad H_G|\psi_0\rangle=E_0|\psi_0\rangle,\qquad E_0=0 \tag{3}$$

where we have denoted $|0\rangle$ as the state of spin-down $|\downarrow\rangle$, and $|1\rangle$ as the state of spin-up $|\uparrow\rangle$. The (vacuum) state $|\psi_0\rangle$ is a state with all spins down.

The second subspace contains N (because $C_N^1=N$) first-excitation states, which have the following forms

$$|\psi_1\rangle^{(k)}=\sum_{m=1}^{N}a_k(m)\phi(m),\qquad H|\psi_1\rangle^{(k)}=E_1^{(k)}|\psi_1\rangle^{(k)},\quad k=1,2,\ldots,N \tag{4}$$

where

$$\phi(m) = |00\cdots 1_m \cdots 0\rangle \tag{5}$$

represents a state in which only the spin on the m-th lattice is up.

The third subspace contains $C_N^2 = N(N-1)/2$ second-excitation states, which have the following forms

$$|\psi_2\rangle^{(k)} = \sum_{m1<m2}^{N} a_k(m1, m2)\phi(m1, m2), \qquad H|\psi_2\rangle^{(k)} = E_2^{(k)}|\psi_2\rangle^{(k)},$$
$$k = 1, 2, \ldots, N(N-1)/2 \tag{6}$$

where

$$\phi(m1, m2) = |\cdots 1_{m1} \cdots 1_{m2} \cdots\rangle \tag{7}$$

represents a state in which only the spins on the $m1$-th and $m2$-th lattices are up.

The fourth subspace contains $C_N^3 = N(N-1)(N-2)/6$ third-excitation states, which have the following forms

$$|\psi_3\rangle^{(k)} = \sum_{m1<m2<m3}^{N} a_k(m1, m2, m3)\phi(m1, m2, m3), \qquad H|\psi_3\rangle^{(k)} = E_3^{(k)}|\psi_3\rangle^{(k)},$$
$$k = 1, 2, \ldots, N(N-1)(N-2)/6 \tag{8}$$

where

$$\phi(m1, m2, m3) = |\cdots 1_{m1} \cdots 1_{m2} \cdots 1_{m3} \cdots\rangle \tag{9}$$

represents a state in which only the spins on the $m1$-th, $m2$-th and $m3$-th lattices are up. Similarly the $(j+1)$-th subspace contains $C_N^j = \frac{N!}{j!(N-j)!}$ states, and the last subspace contains only one state with all spins up.

For the modified Heisenberg XX model as shown in Ref. [21], the coupling strengths are chosen as

$$J_{n,n+1} = \lambda\sqrt{n(N-n)} \tag{10}$$

which are site-dependent (λ is a real number). When we restrict the Hamiltonian H to the second subspace, from the point of view of the matrix language, the Hamiltonian H corresponds to the following $N \times N$ matrix [21]

$$H = \frac{\lambda}{2}\begin{pmatrix} 0 & J_{12} & 0 & 0 & \cdots & 0 \\ J_{12} & 0 & J_{23} & 0 & \cdots & 0 \\ 0 & J_{23} & 0 & J_{34} & \cdots & 0 \\ 0 & 0 & J_{34} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & J_{N-1,N} \\ 0 & 0 & 0 & 0 & J_{N-1,N} & 0 \end{pmatrix}. \tag{11}$$

The coupling strength $J_{n,n+1} = \lambda\sqrt{n(N-n)}$ has definite significance. Under such a selection, the Hamiltonian $H = \lambda S_x$ is proportional to the x-component of angular momentum operator for spin $j = \frac{1}{2}(N-1)$ particle. Therefore, it is easy to get the eigenvalues of H by the ones of λS_z (because the eigenvalues of S_x and S_z are the same).

3 Structure of Energy Spectrum and Some Examples

It is well-known that, for spin-j particle, the operator S_z has $N = 2j + 1$ eigenvalues $\{-j, -j+1, \ldots, j-1, j\}$. From previous section, we know that there are N eigenvalues $E_1^{(k)}$ $(k = 1, 2, \ldots, N)$ in the second subspace, which are the same as the eigenvalues of λS_z, i.e., $E_1^{(k)} \in \lambda\{-\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2}\}$. For convenient, let us denote the set $\lambda\{-j, -j+1, \ldots, j-1, j\}/2$ by $\{v_1, v_2, \ldots, v_{N-1}, v_N\}$, then the energy spectrum has the following interesting structure:

For the first subspace,

$$E_0^{(k)} = v_1 + v_2 + \cdots + v_N = 0 \quad (k = 1). \tag{12}$$

For the second subspace, the N eigenvalues are

$$\begin{aligned} &E_1^{(k)} = v_1 + \cdots + v_{k-1} - v_k + v_{k+1} + \cdots + v_N, \\ &k = 1, 2, \ldots, N \end{aligned} \tag{13}$$

namely, in the front of v_k there is a minus sign, others have positive signs. For the third subspace, the C_N^2 eigenvalues are

$$\begin{aligned} &E_2^{(k1,k2)} = v_1 + \cdots + v_{k1-1} - v_{k1} + v_{k1+1} + \cdots + v_{k2-1} - v_{k2} + v_{k2+1} + \cdots + v_N, \\ &k1 \neq k2 = 1, 2, \ldots, N \end{aligned} \tag{14}$$

namely, v_{k1} and v_{k2} have minus signs, others have positive signs. The similar procedure can be done for the general n-th subspace. In the following, we give some explicit examples for low dimensions $N = 2, 3, \ldots, 6$. For convenient, let us set $\lambda = 2$ in the following.

(i): For $N = 2$, the four eigenvalues and their corresponding eigenstates are

$$\begin{aligned} &E_0 = -1, \qquad E_1 = 1, \qquad E_2 = E_3 = 0. \\ &|\phi_0\rangle = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle), \qquad |\phi_1\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle), \\ &|\phi_2\rangle = |11\rangle, \qquad |\phi_3\rangle = |00\rangle. \end{aligned} \tag{15}$$

(ii): For $N = 3$,

$$\begin{aligned} &E_0 = E_1 = -2, \qquad E_2 = E_3 = 2, \qquad E_4 = E_5 = E_6 = E_7 = 0 \\ &|\phi_0\rangle = \frac{1}{2}(|011\rangle - \sqrt{2}|101\rangle + |110\rangle), \qquad |\phi_1\rangle = \frac{1}{2}(|001\rangle - \sqrt{2}|010\rangle + |100\rangle), \\ &|\phi_2\rangle = \frac{1}{2}(|011\rangle + \sqrt{2}|101\rangle + |110\rangle), \qquad |\phi_3\rangle = \frac{1}{2}(|001\rangle + \sqrt{2}|010\rangle + |100\rangle), \\ &|\phi_4\rangle = |111\rangle, \qquad |\phi_5\rangle = \frac{1}{\sqrt{2}}(-|011\rangle + |110\rangle), \qquad |\phi_6\rangle = \frac{1}{\sqrt{2}}(-|001\rangle + |100\rangle), \\ &|\phi_7\rangle = |000\rangle. \end{aligned} \tag{16}$$

(iii): For $N = 4$,

$$E_0 = -4, \qquad E_1 = 4, \qquad E_2 = E_3 = -3, \qquad E_4 = E_5 = 3, \qquad E_6 = -2, \ E_7 = 2,$$

$$\begin{aligned}
&E_8 = E_9 = -1, \qquad E_{10} = E_{11} = 1, \qquad E_{12} = E_{13} = E_{14} = E_{15} = 0.\\
&|\phi_0\rangle = \frac{1}{4}(|0011\rangle - 2|0101\rangle + \sqrt{3}|0110\rangle + \sqrt{3}|1001\rangle - 2|1010\rangle + |1100\rangle),\\
&|\phi_1\rangle = \frac{1}{4}(|0011\rangle + 2|0101\rangle + \sqrt{3}|0110\rangle + \sqrt{3}|1001\rangle + 2|1010\rangle + |1100\rangle),\\
&|\phi_2\rangle = \frac{1}{2\sqrt{2}}(-|0111\rangle + \sqrt{3}|1011\rangle - \sqrt{3}|1101\rangle + |1110\rangle),\\
&|\phi_3\rangle = \frac{1}{2\sqrt{2}}(-|0001\rangle + \sqrt{3}|0010\rangle - \sqrt{3}|0100\rangle + |1000\rangle),\\
&|\phi_4\rangle = \frac{1}{2\sqrt{2}}(|0111\rangle + \sqrt{3}|1011\rangle + \sqrt{3}|1101\rangle + |1110\rangle),\\
&|\phi_5\rangle = \frac{1}{2\sqrt{2}}(|0001\rangle + \sqrt{3}|0010\rangle + \sqrt{3}|0100\rangle + |1000\rangle), \qquad (17)\\
&|\phi_6\rangle = \frac{1}{2}(-|0011\rangle + |0101\rangle - |1010\rangle + |1100\rangle),\\
&|\phi_7\rangle = \frac{1}{2}(-|0011\rangle - |0101\rangle + |1010\rangle + |1100\rangle),\\
&|\phi_8\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|0111\rangle - |1011\rangle - |1101\rangle + \sqrt{3}|1110\rangle),\\
&|\phi_9\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|0001\rangle - |0010\rangle - |0100\rangle + \sqrt{3}|1000\rangle),\\
&|\phi_{10}\rangle = \frac{1}{2\sqrt{2}}(-\sqrt{3}|0111\rangle - |1011\rangle + |1101\rangle + \sqrt{3}|1110\rangle),\\
&|\phi_{11}\rangle = \frac{1}{2\sqrt{2}}(-\sqrt{3}|0001\rangle - |0010\rangle + |0100\rangle + \sqrt{3}|1000\rangle),\\
&|\phi_{12}\rangle = |1111\rangle, \qquad |\phi_{13}\rangle = \frac{1}{\sqrt{10}}(\sqrt{3}|0011\rangle - 2|0110\rangle + \sqrt{3}|1100\rangle),\\
&|\phi_{14}\rangle = \frac{1}{2\sqrt{10}}(-\sqrt{3}|0011\rangle - 3|0110\rangle + 5|1001\rangle - \sqrt{3}|1100\rangle), \qquad |\phi_{15}\rangle = |0000\rangle.
\end{aligned}$$

(iv): For $N = 5$,

$$\begin{aligned}
&E_0 = E_1 = -6, \qquad E_2 = E_3 = 6, \qquad E_4 = E_5 = E_6 = E_7 = -4,\\
&E_8 = E_9 = E_{10} = E_{11} = 4, \qquad E_{12} = E_{13} = E_{14} = E_{15} = E_{16} = E_{17} = -2,\\
&E_{18} = E_{19} = E_{20} = E_{21} = E_{22} = E_{23} = 2, \qquad (18)\\
&E_{24} = E_{25} = E_{26} = E_{27} = E_{28} = E_{29} = E_{30} = E_{31} = 0.
\end{aligned}$$

$$\begin{aligned}
|\phi_0\rangle = {}& \frac{1}{8}(|00111\rangle - \sqrt{6}|01011\rangle + 3|01101\rangle - 2|01110\rangle + \sqrt{6}|10011\rangle\\
&- 4|10101\rangle + 3|10110\rangle + \sqrt{6}|11001\rangle - \sqrt{6}|11010\rangle + |11100\rangle),
\end{aligned}$$

$$
\begin{aligned}
|\phi_1\rangle = {}& \frac{1}{8}(|00011\rangle - \sqrt{6}|00101\rangle + \sqrt{6}|00110\rangle + 3|01001\rangle - 4|01010\rangle \\
&+\sqrt{6}|01100\rangle - 2|10001\rangle + 3|10010\rangle - \sqrt{6}|10100\rangle + |11000\rangle, \\
|\phi_2\rangle = {}& \frac{1}{8}(|00111\rangle + \sqrt{6}|01011\rangle + 3|01101\rangle + 2|01110\rangle + \sqrt{6}|10011\rangle \\
&+ 4|10101\rangle + 3|10110\rangle + \sqrt{6}|11001\rangle + \sqrt{6}|11010\rangle + |11100\rangle), \\
|\phi_3\rangle = {}& \frac{1}{8}(|00011\rangle + \sqrt{6}|00101\rangle + \sqrt{6}|00110\rangle + 3|01001\rangle + 4|01010\rangle \\
&+ \sqrt{6}|01100\rangle + 2|10001\rangle + 3|10010\rangle + \sqrt{6}|10100\rangle + |11000\rangle, \\
|\phi_4\rangle = {}& \frac{1}{4}(|01111\rangle - 2|10111\rangle + \sqrt{6}|11011\rangle - 2|11101\rangle + |11110\rangle), \\
|\phi_5\rangle = {}& \frac{1}{4\sqrt{2}}(-\sqrt{3}|00111\rangle + 2\sqrt{2}|01011\rangle - \sqrt{3}|01101\rangle - \sqrt{2}|10011\rangle \\
&+ \sqrt{3}|10110\rangle + \sqrt{2}|11001\rangle - 2\sqrt{2}|11010\rangle + \sqrt{3}|11100\rangle), \\
|\phi_6\rangle = {}& \frac{1}{4\sqrt{2}}(-\sqrt{3}|00011\rangle + 2\sqrt{2}|00101\rangle - \sqrt{2}|00110\rangle - \sqrt{3}|01001\rangle + \sqrt{2}|01100\rangle \\
&+ \sqrt{3}|10010\rangle - 2\sqrt{2}|10100\rangle + \sqrt{3}|11000\rangle), \\
|\phi_7\rangle = {}& \frac{1}{4}(|00001\rangle - 2|00010\rangle + \sqrt{6}|00100\rangle - 2|01000\rangle + |10000\rangle), \\
|\phi_8\rangle = {}& \frac{1}{4}(|01111\rangle + 2|10111\rangle + \sqrt{6}|11011\rangle + 2|11101\rangle + |11110\rangle), \\
|\phi_9\rangle = {}& \frac{1}{4\sqrt{2}}(-\sqrt{3}|00111\rangle - 2\sqrt{2}|01011\rangle - \sqrt{3}|01101\rangle - \sqrt{2}|10011\rangle \\
&+ \sqrt{3}|10110\rangle + \sqrt{2}|11001\rangle + 2\sqrt{2}|11010\rangle + \sqrt{3}|11100\rangle), \\
|\phi_{10}\rangle = {}& \frac{1}{4\sqrt{2}}(-\sqrt{3}|00011\rangle - 2\sqrt{2}|00101\rangle - \sqrt{2}|00110\rangle - \sqrt{3}|01001\rangle + \sqrt{2}|01100\rangle \\
&+ \sqrt{3}|10010\rangle + 2\sqrt{2}|10100\rangle + \sqrt{3}|11000\rangle), \\
|\phi_{11}\rangle = {}& \frac{1}{4}(|00001\rangle + 2|00010\rangle + \sqrt{6}|00100\rangle + 2|01000\rangle + |10000\rangle), \\
|\phi_{12}\rangle = {}& \frac{1}{2}(-|01111\rangle + |10111\rangle - |11101\rangle + |11110\rangle), \\
|\phi_{13}\rangle = {}& \frac{\sqrt{15}}{8}(|00111\rangle - \sqrt{\frac{2}{3}}|01011\rangle - \frac{1}{5}|01101\rangle + \frac{2}{5}|01110\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|10011\rangle \\
&+ \frac{4}{5}|10101\rangle - \frac{1}{5}|10110\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|11001\rangle - \sqrt{\frac{2}{3}}|11010\rangle + |11100\rangle), \qquad (19)
\end{aligned}
$$

$$|\phi_{14}\rangle = \frac{1}{2\sqrt{5}}(-\sqrt{2}|01101\rangle + 2\sqrt{2}|01110\rangle + \sqrt{3}|10011\rangle - \sqrt{2}|10101\rangle - \sqrt{2}|10110\rangle + \sqrt{3}|11001\rangle),$$

$$|\phi_{15}\rangle = \frac{\sqrt{15}}{8}\bigg(|00011\rangle - \sqrt{\frac{2}{3}}|00101\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|00110\rangle - \frac{1}{5}|01001\rangle + \frac{4}{5}|01010\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|01100\rangle + \frac{2}{5}|10001\rangle - \frac{1}{5}|10010\rangle - \sqrt{\frac{2}{3}}|10100\rangle + |11000\rangle\bigg),$$

$$|\phi_{16}\rangle = \frac{1}{\sqrt{10}}\bigg(-\sqrt{\frac{3}{2}}|00110\rangle + |01001\rangle + |01010\rangle - \sqrt{\frac{3}{2}}|01100\rangle - 2|10001\rangle + |10010\rangle\bigg),$$

$$|\phi_{17}\rangle = \frac{1}{2}(-|00001\rangle + |00010\rangle - |01000\rangle + |10000\rangle),$$

$$|\phi_{18}\rangle = \frac{1}{2}(-|01111\rangle - |10111\rangle + |11101\rangle + |11110\rangle),$$

$$|\phi_{19}\rangle = \frac{\sqrt{15}}{8}\bigg(|00111\rangle + \sqrt{\frac{2}{3}}|01011\rangle - \frac{1}{5}|01101\rangle - \frac{2}{5}|01110\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|10011\rangle - \frac{4}{5}|10101\rangle - \frac{1}{5}|10110\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|11001\rangle + \sqrt{\frac{2}{3}}|11010\rangle + |11100\rangle\bigg),$$

$$|\phi_{20}\rangle = \frac{1}{2\sqrt{5}}(-\sqrt{2}|01101\rangle - 2\sqrt{2}|01110\rangle + \sqrt{3}|10011\rangle + \sqrt{2}|10101\rangle - \sqrt{2}|10110\rangle + \sqrt{3}|11001\rangle),$$

$$|\phi_{21}\rangle = \frac{\sqrt{15}}{8}\bigg(|00011\rangle + \sqrt{\frac{2}{3}}|00101\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|00110\rangle - \frac{1}{5}|01001\rangle - \frac{4}{5}|01010\rangle - \frac{\sqrt{\frac{2}{3}}}{5}|01100\rangle - \frac{2}{5}|10001\rangle - \frac{1}{5}|10010\rangle + \sqrt{\frac{2}{3}}|10100\rangle + |11000\rangle\bigg),$$

$$|\phi_{22}\rangle = \frac{1}{\sqrt{10}}\bigg(-\sqrt{\frac{3}{2}}|00110\rangle + |01001\rangle - |01010\rangle - \sqrt{\frac{3}{2}}|01100\rangle + 2|10001\rangle + |10010\rangle\bigg),$$

$$|\phi_{23}\rangle = \frac{1}{2}(-|00001\rangle - |00010\rangle + |01000\rangle + |10000\rangle),$$

$$|\phi_{24}\rangle = |11111\rangle,$$

$$|\phi_{25}\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|01111\rangle - \sqrt{2}|11011\rangle + \sqrt{3}|11110\rangle),$$

$$|\phi_{26}\rangle = \frac{\sqrt{5}}{4}(-|00111\rangle + \frac{3}{5}|01101\rangle + \frac{\sqrt{6}}{5}|10011\rangle - \frac{3}{5}|10110\rangle - \frac{\sqrt{6}}{5}|11001\rangle + |11100\rangle),$$

$$|\phi_{27}\rangle = \frac{1}{\sqrt{10}}(\sqrt{2}|01101\rangle - \sqrt{3}|10011\rangle - \sqrt{2}|10110\rangle + \sqrt{3}|11001\rangle),$$

$$|\phi_{28}\rangle = \frac{\sqrt{5}}{4}\left(-|00011\rangle + \frac{\sqrt{6}}{5}|00110\rangle + \frac{3}{5}|01001\rangle - \frac{\sqrt{6}}{5}|01100\rangle - \frac{3}{5}|10010\rangle + |11000\rangle\right),$$
$$|\phi_{29}\rangle = \frac{1}{\sqrt{5}}\left(\sqrt{\frac{3}{2}}|00110\rangle - |01001\rangle - \sqrt{\frac{3}{2}}|01100\rangle + |10010\rangle\right),$$
$$|\phi_{30}\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|00001\rangle - \sqrt{2}|00100\rangle + \sqrt{3}|10000\rangle),$$
$$|\phi_{31}\rangle = |00000\rangle.$$

(v): For $N = 6$, the eigenvalues are

$$\begin{aligned}
&E_0 = -9, \quad E_1 = 9, \quad E_2 = E_3 = -8, \quad E_4 = E_5 = 8,\\
&E_6 = -7, \quad E_7 = 7, \quad E_8 = E_9 = -6, \quad E_{10} = E_{11} = 6,\\
&E_{12} = E_{13} = E_{14} = E_{15} = -5, \quad E_{16} = E_{17} = E_{18} = E_{19} = 5,\\
&E_{20} = E_{21} = E_{22} = E_{23} = -4, \quad E_{24} = E_{25} = E_{26} = E_{27} = 4,\\
&E_{28} = E_{29} = E_{30} = E_{31} = E_{32} = -3, \quad E_{33} = E_{34} = E35 = E_{36} = E_{37} = 3,\\
&E_{38} = E_{39} = E_{40} = E_{41} = -2,\\
&E_{42} = E_{43} = E_{44} = E_{45} = 2, \quad E_{46} = E_{47} = E_{48} = E_{49} = E_{50} = -1,\\
&E_{51} = E_{52} = E_{53} = E_{54} = E_{55} = 1,\\
&E_{56} = E_{57} = E_{58} = E_{59} = E_{60} = E_{61} = E_{62} = E_{63} = 0
\end{aligned} \tag{20}$$

the corresponding 64 eigenstates can also be calculated, however, to save the length of the paper, here we only list the eigenstates corresponding the lowest and highest eigenvalues

$$\begin{aligned}
|\phi\rangle_{\text{lowest}} = \{&0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 3, 0, -3\sqrt{2}, \sqrt{10}, 0, 0, 0, 0, -3\sqrt{2}, 0, 8,\\
&-3\sqrt{5}, 0, 0, -6, 3\sqrt{5}, 0, -\sqrt{10}, 0, 0, 0, 0, 0, 0, \sqrt{10}, 0, -3\sqrt{5}, 6, 0, 0, 3\sqrt{5},\\
&-8, 0, 3\sqrt{2}, 0, 0, 0, 0, -\sqrt{10}, 3\sqrt{2}, 0, -3, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}/16\\
|\phi\rangle_{\text{highest}} = \{&0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 3, 0, 3\sqrt{2}, \sqrt{10}, 0, 0, 0, 0, 3\sqrt{2}, 0, 8, 3\sqrt{5}, 0, 0, 6,\\
&3\sqrt{5}, 0, \sqrt{10}, 0, 0, 0, 0, 0, 0, \sqrt{10}, 0, 3\sqrt{5}, 6, 0, 0, 3\sqrt{5}, 8, 0, 3\sqrt{2}, 0, 0, 0, 0,\\
&\sqrt{10}, 3\sqrt{2}, 0, 3, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}/16,
\end{aligned} \tag{21}$$

where the first entry is the coefficient of the state $|000000\rangle$ and so forth.

Furthermore, it is worth to mention that, for $N =$ odd, the highest (or lowest) eigenvalue is 2-fold degenerate, while for $N =$ even, the highest (or lowest) eigenvalue is not degenerate. The analytic expression for the highest as well as the lowest eigenvalue is

$$\begin{aligned}
&E_{\min} = k - k^2, \quad E_{\max} = k^2 - k, \quad \text{for } N = 2k - 1;\\
&E_{\min} = -k^2, \quad E_{\max} = k^2, \quad \text{for } N = 2k\ (k = 2, 3, \ldots)
\end{aligned} \tag{22}$$

4 Discussion and Conclusion

In this paper, we show the interesting structure of energy spectrum for the modified Heisenberg XX model, explicit examples for low dimensions $N = 2, 3, \ldots, 6$ are also provided. The modified model has an important application in quantum information, such as the perfect state transfer. Concepts such as entanglement and its implications concerning the non-existence of a local realism in quantum mechanics have a more fundamental role in quantum mechanics. The issue of "locality" as well as notion of quantum measurements has given rise to some of the recent and modern interpretations of quantum mechanics as well as a better understanding of quantum phenomena. Recently, an interesting topic is to investigate the nonlocality of a spin chain through thermal entanglement. When spin chains are subjected to environmental disturbance, they inevitably becomes thermal equilibrium states. The thermal state of a system at finite temperature T is given by the Gibb's density operator $\rho(T) = \exp(-H/kT)/Z$, where $Z = \mathrm{Tr}[\exp(-H/kT)]$ is the partition function, H is the system Hamiltonian and k is the Boltzmann constant. Below the critical temperature (threshold temperature) T_0, the thermal state is really nonlocal. In terms of correlation function defined by

$$Q_{i_1 i_2,\ldots,i_N} = \mathrm{Tr}[\rho(\hat{n}_{i_1} \cdot \vec{\sigma}) \otimes (\hat{n}_{i_2} \cdot \vec{\sigma}) \otimes \cdots \otimes (\hat{n}_{i_N} \cdot \vec{\sigma})] \tag{23}$$

this nonlocality property can be shown by the violation of Zukowski–Brukner inequality for N-qubit [23]. Ref. [22] has studied the nonlocality of the modified XX model for $N = 4$, interested reader can test nonlocality for higher dimensions of N.

Acknowledgement C. Wu acknowledges financial support from the Singapore Millennium Foundation.

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